

# Differential Equations in Science and Engineering

Exam January 29, 2025, 11:45-13:45.

This exam has 5 problems. The maximum score is 9 points + 1 (free) = 10. All answers must be supported by work or reasoning.

1. The Kermack-McKendrick model is used to describe epidemics, such as black death or cholera. This model assumes the population can be separated into three groups. One is the population  $I(t)$  that is ill at time  $t$ , another is the population  $S(t)$  that is susceptible to the disease, and the third is the population  $R(t)$  of individuals that have recovered. The model is given by

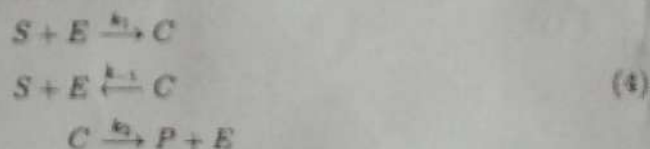
$$\frac{dS}{dt} = -k_1 SI, \quad (1)$$

$$\frac{dI}{dt} = -k_2 I + k_1 SI, \quad (2)$$

$$\frac{dR}{dt} = k_2 I. \quad (3)$$

- (a) (0.5 pts) The right-hand side of Eq. (1) does not depend on  $R$  and that of Eq. (3) does not depend on  $S$ , because certain model assumption have been made.
- Formulate a model assumption that leads to an equation describing the time-rate change of  $S$  without using  $R$ .
  - Which model assumption results in a right-hand side of Eq. (3) that does not depend on  $S$ ?
- (b) (0.5 pts) Determine a conserved quantity of the Kermack-McKendrick model and show that it is conserved.

2. We consider the following prototype example for an enzyme-catalyzed reaction:



In this reaction,  $S$  is the substance that is transformed by the reaction,  $E$  is the enzyme that facilitates the conversion,  $C$  is an intermediate complex, and  $P$  is the final product produced by the reaction.

- (a) (0.5 pts) Draw the reaction network.
- (b) (2 pts) Derive the stoichiometric net coefficients, the reaction rates, the production rates, and the corresponding system of ODEs that describes the dynamics of the species' concentrations denoted by  $n_S, n_E, n_C, n_P$ .
- (c) (0.5 pts) For initial conditions, it is assumed that we start with  $S$  and  $E$  and no complex and product, i.e.,  $n_S(0) = s_0, n_E(0) = e_0, n_C(0) = 0, n_P(0) = 0$ , where  $s_0$  and  $e_0$  are given. Two useful conservation laws for this reaction are  $\frac{d}{dt}(n_E + n_C) = 0$  and  $\frac{d}{dt}(n_S + n_C + n_P) = 0$ . Reduce the ODE system you determined in part (b) of this exercise to a system consisting of only two (scalar) differential equations.

Exercises continue on the other side

3. We consider the dynamical system

$$\begin{aligned}\frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -y - \alpha x(1-x),\end{aligned}$$

where  $\alpha$  is a nonzero constant;  $x$  and  $y$  are functions of time  $t$ .

- (a) (0.2 pts) Compute two steady states of this system.
- (b) (0.8 pts) Determine the eigenvalues of the Jacobian at these steady states.
- (c) (1.0 pts) Discuss the stability of these steady states.

4. We consider the scalar partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^4 u}{\partial x^4}, \quad (5)$$

with constant velocity  $c \in \mathbb{R}$  and  $c \neq 0$ .

We want to perform a linear stability analysis of equation (5) using the wave ansatz

$$u(t, x) = u_0 \cdot e^{i(kx - \omega t)}, \quad (6)$$

for wave number  $k \in \mathbb{R}$ , wave frequencies  $\omega \in \mathbb{C}$  and amplitude  $u_0 \in \mathbb{R}$ .

- (a) (1.0 pts) What wave frequencies  $\omega$  in (6) lead to a non-increasing, i.e. stable, wave in time?
- (b) (1.0 pts) Insert the wave ansatz (6) into the wave equation (5) to show that it is unstable for all nonzero  $c \in \mathbb{R}$ .

5. The system

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \quad \frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2)}{\partial x} = 0,$$

can be cast in the form

$$\partial_t \begin{pmatrix} \rho \\ v \end{pmatrix} = A(\rho, v) \partial_x \begin{pmatrix} \rho \\ v \end{pmatrix},$$

where  $A(\rho, v)$  is a  $2 \times 2$ -matrix with entries depending on  $\rho$  and  $v$ .

- (a) (0.8 pts) Determine  $A(\rho, v)$ .
- (b) (0.6 pts) Compute the eigenvalues of  $A(\rho, v)$ . Show that they are real if  $v$  is real.
- (c) (0.6 pts) Compute the eigenvector(s) of  $A(\rho, v)$ . Is  $A(\rho, v)$  diagonalizable?