Differential Equations in Science and Engineering

Exam January 29, 2025, 11.45-13.45.

This exam has 5 problems. The maximum score is 9 points + 1 (free) = 10. All answers must be supported by work or reasoning.

1. The Kermack-McKendrick model is used to describe epidemics, such as black death or cholera. This model assumes the population can be separated into three groups. One is the population $I(t)$ that is ill at time t, another is the population $S(t)$ that is susceptible to the disease, and the third is the population $R(t)$ of individuals that have recovered. The model is given by

$$\frac{dS}{dt} = -k_1 S I, \tag{1}$$
$$\frac{dI}{dt} = -k_2 I + k_1 S I, \tag{2}$$
$$\frac{dR}{dt} = k_2 I. \tag{3}$$

   (a) (0.5 pts) The right-hand side of Eq. (1) does not depend on R and that of Eq. (3) does not depend on S, because certain model assumption have been made.

      i. Formulate a model assumption that leads to an equation describing the time-rate change of S without using R.

      ii. Which model assumption results in a right-hand side of Eq. (3) that does not depend on S?

   (b) (0.5 pts) Determine a conserved quantity of the Kermack-McKendrick model and show that it is conserved.

2. We consider the following prototype example for an enzyme-catalyzed reaction:

$$\begin{aligned} S + E &\xrightarrow{k_1} C \\ S + E &\xleftarrow{k_{-1}} C \\ C &\xrightarrow{k_2} P + E \end{aligned} \tag{4}$$

In this reaction, S is the substance that is transformed by the reaction, E is the enzyme that facilitates the conversion, C is an intermediate complex, and P is the final product produced by the reaction.

   (a) (0.5 pts) Draw the reaction network.

   (b) (2 pts) Derive the stoichiometric net coefficients, the reaction rates, the production rates, and the corresponding system of ODEs that describes the dynamics of the species' concentrations denoted by n_S, n_E, n_C, n_P.

   (c) (0.5 pts) For initial conditions, it is assumed that we start with S and E and no complex and product, i.e., $n_S(0) = s_0$, $n_E(0) = e_0$, $n_C(0) = 0$, $n_P(0) = 0$, where s_0 and e_0 are given. Two useful conservation laws for this reaction are $\frac{d}{dt}(n_E + n_C) = 0$ and $\frac{d}{dt}(n_S + n_C + n_P) = 0$. Reduce the ODE system you determined in part (b) of this exercise to a system consisting of only two (scalar) differential equations.

Excercises continue on the other side

3. We consider the dynamical system

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -y - \alpha x(1-x),\end{aligned}$$

where α is a nonzero constant; x and y are functions of time t.

(a) (0.2 pts) Compute two steady states of this system.

(b) (0.8 pts) Determine the eigenvalues of the Jacobian at these steady states.

(c) (1.0 pts) Discuss the stability of these steady states.

4. We consider the scalar partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^4 u}{\partial x^4}, \tag{5}$$

with constant velocity $c \in \mathbb{R}$ and $c \neq 0$.

We want to perform a linear stability analysis of equation (5) using the wave ansatz

$$u(t,x) = u_0 \cdot e^{i(kx - \omega t)}, \tag{6}$$

for wave number $k \in \mathbb{R}$, wave frequencies $\omega \in \mathbb{C}$ and amplitude $u_0 \in \mathbb{R}$.

(a) (1.0 pts) What wave frequencies ω in (6) lead to a non-increasing, i.e. stable, wave in time?

(b) (1.0 pts) Insert the wave ansatz (6) into the wave equation (5) to show that it is unstable for all nonzero $c \in \mathbb{R}$.

5. The system

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \qquad \frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2)}{\partial x} = 0,$$

can be cast in the form

$$\partial_t \begin{pmatrix}\rho\\ v\end{pmatrix} = A(\rho, v)\, \partial_x \begin{pmatrix}\rho\\ v\end{pmatrix},$$

where $A(\rho, v)$ is a 2×2-matrix with entries depending on ρ and v.

(a) (0.8 pts) Determine $A(\rho, v)$.

(b) (0.6 pts) Compute the eigenvalues of $A(\rho, v)$. Show that they are real if v is real.

(c) (0.6 pts) Compute the eigenvector(s) of $A(\rho, v)$. Is $A(\rho, v)$ diagonalizable?